ON SOME PROPERTIES OF FUNCTIONS FROM A BESOV-MORREY TYPE SPACES

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Abstract. In this paper it is constructed a new Besov-Morrey type spaces. Utilizing integral representation of generalized derivatives of functions defined on n-dimensional domains satisfying flexible φ-horn condition an embedding theorem is proved. Also, it is proved that the generalized derivatives of functions from this spaces satisfies the generalized Hölder condition.

In this paper we introduce a Besov-Morrey type spaces

$$B^l_{p,\theta,\varphi,\beta}(G_\varphi)$$

and studied differential and differential-difference properties of functions from this spaces. Note that the spaces with parameters based on isotropic Sobolev space, while some I.Ross private values of indexes for the first time studied in the works of Ch.Morrey [8]. These results further developed and compiled in the works of V.P. Il'yin [6], Yu.V. Netrusov [13], V.S. Guliyev [3], Y.Sawano[15], V.Kokilashvili, A. Meskhi and H. Rafeiro [7], I. Ross[14] and [1, 10, 11, 12] and others.

Let G be a domain in $\mathbb{R}^n$ and let $\varphi(t) = (\varphi_1(t), \ldots, \varphi_n(t))$, $\varphi_j(t) > 0$, $\varphi'_j(t) > 0$ $(t > 0)$ is continuously differentiable functions. Assume that $\lim\limits_{t\to+0} \varphi_j(t) = 0$ and $\lim\limits_{t\to+\infty} \varphi_j(t) = P_j$, where $0 < P_j \leq \infty$, $(j = 1, 2, \ldots, n)$. We denote the set of such vector-functions φ by A. We assume that $|\varphi([t]_1)|^{-\beta} = \prod\limits_{j=1}^{n} (\varphi_j([t]_1))^{-\beta_j}$, $\beta_j \in [0, 1]$ $(j = 1, 2, \ldots, n)$ and $[t]_1 = \min\{1, t\}$.

For any $x \in \mathbb{R}^n$ we put

$$\begin{aligned} G_{\varphi(t)}(x) &= G \cap I_{\varphi(t)}(x) \\ &= G \cap \left\{ y : |y_j - x_j| < \frac{1}{2}\varphi_j(t), (j = 1, 2, ..., n) \right\}, \end{aligned}$$

$$\Delta_i^{m_i}\left(\varphi_i(t), G_{\varphi(t)}\right) f(x) = \begin{cases} \Delta_i^{m_i}(\varphi_i(t)) f(x), & for \ \ [x,\, x + m_i\varphi_i(t)e_i] \subset G_{\varphi(t)}, \\ 0, & for \ \ [x,\, x + m_i\varphi_i(t)e_i] \not\subseteq G_{\varphi(t)} \end{cases}$$

$$\Delta_i^{m_i}(\varphi_i(t)) f(x) = \sum_{j=0}^{m_i} (-1)^{m_i - j} C^j_{m_i} f(x + j\varphi_i(t)e_i),\ e_i = \Bigg(\underbrace{0, \ldots, 0}_{i-1}, 1, 0, \ldots, 0\Bigg),$$

where $m_i \in N$.

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Definition 0.1. Let $l \in (0,\infty)^n$, $k_i \in N_0$, $1 \le \theta \le \infty$ and $1 \le p < \infty$. A space of the form $B^l_{p,\theta,\varphi,\beta}(G_\varphi)$ is defined as a linear normed space of functions f, on $G \subset \mathbb{R}^n$, with the finite norm $(m_i > l_i - k_i > 0\ (i = 1,...,n))$:

$$\|f\|_{B^l_{p,\theta,\varphi,\beta}(G_\varphi)} = \|f\|_{p,\varphi,\beta;G} +$$
$$+\sum_{i=0}^{n}\left\{\int_0^{t_0}\left[\frac{\left\|\Delta_i^{m_i}(\varphi_i(t),G_{\varphi(t)})D_i^{k_i}f\right\|_{p,\varphi,\beta}}{(\varphi_i(t))^{(l_i-k_i)}}\right]^\theta \frac{d\varphi_i(t)}{\varphi_i(t)}\right\}^{\frac{1}{\theta}}, \tag{0.1}$$

where $t_0 > 0$ is a fixed number and

$$\|f\|_{p,\varphi,\beta;G} = \|f\|_{L_{p,\varphi,\beta}(G)} = \sup_{\substack{x\in G,\\ t>0}}\left(|\varphi([t]_1)|^{-\beta}\|f\|_{p,G_{\varphi(t)}(x)}\right). \tag{0.2}$$

Definition 0.2. An open set $G \subset \mathbb{R}^n$ is said to satisfy condition of flexible φ-horn, if for some $\theta \in (0,1]^n$, $T \in (0,\infty)$ for any $x \in G$ there exists a vector-function

$$\rho(\varphi(t),x) = (\rho_1(\varphi_1(t),x),...,\rho_n(\varphi_n(t),x)),\quad 0 \le t \le T$$

with the following properties:

1) for all $j = 1,2,...,n$, $\rho(\varphi_j(t),x)$ is absolutely continuous on $[0,T]$, $\left|\rho_j'(\varphi_j(t),x)\right| \le 1$ for almost all $t \in [0,T]$,
2) $\rho_j(0,x) = 0;\ x + V(x,\theta) = x + \bigcup_{0\le t\le T}[\rho(\varphi(t),x) + \varphi(t)\theta I] \subset G.$

In particular, $\varphi(t) = t^\lambda$, $\left(t^\lambda = \left(t^{\lambda_1}, t^{\lambda_2},...,t^{\lambda_n}\right)\right)$ is the set $V(x,\theta)$ and $x + V(x,\theta)$ will be said to be a set of flexible λ-horn introduced in [2].

Let for any $t > 0$, there exists a positive constant $C > 0$ such that $|\varphi([t]_1)| \le C$. Then the embeddings

$$L_{p,\varphi,\beta}(G) \hookrightarrow L_p(G), \qquad B^l_{p,\theta,\varphi,\beta}(G_\varphi) \hookrightarrow B^l_{p,\theta}(G_\varphi),$$

hold, i.e

$$\|f\|_{p,G} \le c\|f\|_{p,\varphi,\beta;G}, \qquad \|f\|_{B^l_{p,\theta}(G_\varphi)} \le c\|f\|_{B^l_{p,\theta,\varphi,\beta}(G_\varphi)}, \tag{0.3}$$

where

$$\|f\|_{B^l_{p,\theta}(G_\varphi)} = \|f\|_{p,G} + \sum_{i=0}^{n}\left\{\int_0^{t_0}\left[\frac{\left\|\Delta_i^{m_i}(\varphi_i(t),G_{\varphi(t)})D_i^{k_i}f\right\|_p}{(\varphi_i(t))^{(l_i-k_i)}}\right]^\theta \frac{d\varphi_i(t)}{\varphi_i(t)}\right\}^{\frac{1}{\theta}}.$$

The space $B^l_{p,\theta,\varphi,\beta}(G_\varphi)$ in the case when $\varphi_j(t) = t^{\varkappa_j}$,$\beta_j = \frac{a}{p}$ coincides with the space $B^l_{p,\theta,a,\varkappa}(G_\varphi)$ studied by Yu.V. Netrusov [13], in the case when $\beta_j = 0(j = 1,2,\dots,n)$, coincides with the space $B^l_{p,\theta}(G_\varphi)$, while in the case when $\theta = \infty$ coincides with the space $H^l_{p,\varphi,\beta}(G_\varphi)$ studied in [1]. Spaces with such parameters with different norms were introduced and studied in the papers [4, 5, 9].

It should be noted that the spaces $L_{p,\varphi,\beta}(G)$ and $B^l_{p,\theta,\varphi,\beta}(G_\varphi)$ are complete.

Furthermore, in the case when $G \subset \mathbb{R}^n$ is a bounded domain, $p \le q$, $\varphi(t) \le \psi(t)$ $(t > 0, \psi(t) \in A)$, there exists a constant $C_1 > 0$ such that for any $t \in (0,1)$, $|\psi(t)|^{\beta_1} \le C_1|\varphi(t)|^\beta$, then $L_{q,\psi,\beta_1}(G) \hookrightarrow L_{p,\varphi,\beta}(G)$, i.e. there exists a constant $C_2 > 0$

$$\|f\|_{p,\varphi,\beta;G} \le C_2\|f\|_{q,\psi,\beta_1;G}.$$

It should be noted that the space $B^l_{p,\theta,\varphi,\beta}(G_\varphi)$ preserves all the properties of Nikolski-Morrey type space $H^l_{p,\varphi,\beta}(G_\varphi)$ studied in [1].

Theorem 0.3. *Let $1 \le p < \infty$, $1 \le \theta \le \infty$, $f \in B^l_{p,\theta}(G_\varphi)$ and $\varphi \in A$. Then we can construct the sequence $h_s = h_s(x)$ $(s = 1, 2, \ldots)$ of infinitely differentiable finite in $\mathbb{R}^n$ functions for which*

$$\lim_{s\to\infty} \|f - h_s\|_{B^l_{p,\theta}(G_\varphi)} = 0. \tag{0.4}$$

Prove two theorems on the properties of the functions from the space $B^l_{p,\theta,\varphi,\beta}(G_\varphi)$.

Theorem 0.4. *Let $G \subset \mathbb{R}^n$ satisfy the condition of flexible φ-horn [1], $1 \le p \le q \le \infty$, $\nu = (\nu_1, \nu_2, .., \nu_n)$, $\nu_j \ge 0$ integer $j = 1, 2, ..., n$,$1 \le \theta \le \infty$; $Q^i_T < \infty$ $(i = 1, 2, ..., n)$ and let $f \in B^l_{p,\theta,\varphi,\beta}(G_\varphi)$. Then the following embedding hold*

$$D^\nu : B^l_{p,\theta,\varphi,\beta}(G_\varphi) \hookrightarrow L_{q,\psi,\beta^1}(G)$$

i.e. for $f \in B^l_{p,\theta,\varphi,\beta}(G_\varphi)$ there exists a generalized derivative $D^\nu f$ in G and the following inequalities are true

$$\|D^\nu f\|_{q,G} \le C^1 \Bigg(B(T)\|f\|_{p,\varphi,\beta;G}$$

$$+\sum_{i=1}^n Q^i_T \left\{ \int_0^{t_0} \left[\frac{\|\Delta_i^{m_i}\left(\varphi_i(t), G_{\varphi(t)}\right) f\|_{p,\varphi,\beta}}{(\varphi_i(t))^{l_i}} \right]^\theta \frac{d\varphi_i(t)}{\varphi_i(t)} \right\}^{\frac{1}{\theta}} \Bigg),$$

$$\|D^\nu f\|_{q,\psi,\beta^1;G} \le C^2 \|f\|_{B^l_{p,\theta,\varphi,\beta}(G_\varphi)},\ p \le q < \infty, \tag{0.5}$$

In particular, if

$$Q^i_{T,0} = \int_0^T \prod_{j=1}^n (\varphi_j(t))^{-\nu_j-(1-\beta_j p)\frac{1}{p}} \frac{\varphi_i'(t)}{(\varphi_i(t))^{1-l_i}} dt < \infty, (i = 1, 2, \ldots, n), \tag{0.6}$$

then $D^\nu f(x)$ is continuous on G, and

$$\sup_{x\in G} |D^\nu f(x)| \le C^1 \Bigg(B(t)\|f\|_{p,\varphi,\beta;G}$$

$$+\sum_{i=1}^n Q^i_{T,0} \left\{ \int_0^{t_0} \left[\frac{\|\Delta_i^{m_i}\left(\varphi_i(t), G_{\varphi(t)}\right) f\|_{p,\varphi,\beta}}{(\varphi_i(t))^{l_i}} \right]^\theta \frac{d\varphi_i(t)}{\varphi_i(t)} \right\}^{\frac{1}{\theta}} \Bigg)$$

$0 < T \le \min\{1, t_0\}$ is a fixed number, C^1, C^2 are the constants independent of f, C^1 are independent also on T.

Let γ be an n-dimensional vector.

Theorem 0.5. *Let all the conditions of Theorem 3.1 be satisfied. Then for $Q^i_T < \infty$ $(i = 1, 2, ..., n)$ the generalized derivative $D^\nu f$ satisfies on G the generalized Hölder condition, i.e. the following inequality is valid:*

$$\|\Delta(\gamma, G)\, D^\nu f\|_{q,G} \le C\|f\|_{B^l_{p,\theta,\varphi,\beta}(G_\varphi)} \cdot |h(|\gamma|, \varphi; T)|, \tag{0.7}$$

where C - is a constant independent of f, $|\gamma|$ and T. In particular, if $Q^i_{T,0} < \infty$, $(i = 1, 2, \dots, n)$, then

$$\sup_{x\in G} |\Delta(\gamma, G) D^\nu f(x)| \le C\|f\|_{B^l_{p,\theta,\varphi,\beta}(G_\varphi)} \cdot |h_0(|\gamma|, \varphi, T)|, \tag{0.8}$$

where

$$h(|\gamma|, \varphi, T) = \max_i \left\{|\gamma|, Q^i_{|\gamma|}, Q^i_{|\gamma|,T}\right\}$$

$$\left(h_0(|\gamma|, \varphi, T) = \max_i \left\{|\gamma|, Q^i_{|\gamma|,0}, Q^i_{|\gamma|,T,0}\right\}\right)$$

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